Indian Statistical Institute
Second Semester 2004-2005
Mid Semestral Exam
B.Math I Year
Analysis II

Time: 3 hrs | Date:10-03-05 | [Max marks : 35]

1. Let (A, d), (B, e) be metric spaces satisfying Bolzano-Weirstrass property. On $Y = A \times B$ define a metric m by

$$m((a_1,\ b_1),\ (a_2,\ b_2)) = [d^2(a_1,\ a_2) + e^2(b_1,\ b_2)]^{\frac{1}{2}}$$

Show that $(Y,\ m)$ has the Bolzano Weirstrass property. You can assume that m is a metric. [2]

2. Let $f : (X,\ d) \to (Y,\ m)$ be uniformly continuous. Show that f maps Cauchy sequences into cauchy sequences. [2]

3. Give an example of a C^1 function $\underset{\sim}{f} : [0,1] \to R^2$ such that $\underset{\sim}{f}(0) = \underset{\sim}{f}(1)$, but for each t in $[0,1]$ $\underset{\sim}{f}'(t) \neq (0,\ 0)$ and prove your claim. [2]

4. By using Lagrange method of multipliers find the distance between the two curves C, L where

$$\begin{aligned} C &= \{(x, y) : \quad x^2 + y^2 = 1\} \text{ and} \\ L &= \{(x, y) : x - y = 10\} \end{aligned}$$

[8]

5. Show that the function

$$f(x, y) = 2x + 4y - x^2 y^4$$

has a critical point, but no local maxima or minima. [4]

6. Let $G(x, y) = (x^2+y^2)\sin(x^2+y^2)^{-1}$ for $(x, y) \neq (0, 0)$, and $G(0, 0) = 0$. Note: that $G(x, y) = G(y, x)$

(a) Find $\partial G/\partial x$, $\partial G/\partial y$ on $R^2 \setminus (0, 0)$. [2]

(b) Show that both $\partial G/\partial x$ and $\partial G/\partial y$ are continuous on $R^2 - (0, 0)$. [1]

(c) Show that G has total derivative on $R^2 \setminus (0, 0)$. [1]

(d) Show that $\frac{\partial G}{\partial x}(0, 0) = 0 = \frac{\partial G}{\partial y}(0, 0)$. [1]

(e) Show that G has total derivative at $(0, 0)$ also. [2]

(f) Show that $\partial G/\partial x$ is not continuous at $(0, 0)$. [2]

7. Let $f : R^2 \to R$ be given by $f(x, y) = xy.\ \frac{x^2-y^2}{x^2+y^2}$ on $R^2 \setminus (0, 0), f(0, 0) = 0$. Show that both $\frac{\partial^2 f}{\partial x \partial y}(0, 0),\ \ \frac{\partial^2 f}{\partial y \partial x}(0, 0)$ exist and are different. [4]

8. Let $\phi : R \to R$ be given by $\phi(x) = \exp[-1/x]$ for $x > 0, \quad 0$ for $x \leq 0$. Assume that ϕ is C^∞.

   a) Let $-\infty < a_1 < a_2 < b_1 < b_2 < \infty$. Construct a C^∞ function Ψ such that $\Psi = 1$ on $(a_2,\ b_1), 0$ outside $[a_1\ , b_2],\quad \Psi \geq 0$. [4]

   b) Let $0 < r_1 < r_2$. Find a C^∞ function $g : R^2 \to R$ such that

$$\begin{aligned} g(\underset{\sim}{x}) &= 1 \quad \text{for } ||\underset{\sim}{x}|| \leq r_1 \\ &= 0 \quad \text{for } ||\underset{\sim}{x}|| \geq r_2 \end{aligned}$$

   Hint: $f : R^2 \to R$ given by $f(x_1,\ x_2) = x_1^2 + x_2^2$ is a C^∞ function. [1]